Math 33A Worksheet Week 6 Solutions

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Exercise 1. Let $A : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\operatorname{Im} A$. Notice that $\dim \ker A + \dim \operatorname{Im} A = 4$.

RREF:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so solutions to $Ax = 0$ are given by $x_1 = -2x_2 + x_3 - 3x_4$, x_2, x_3, x_4 free. So a basis for $\ker A$ is given by

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since only first column has a leading one in RREF, the first column $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ forms a basis for $\operatorname{Im} A$.

Exercise 2. True or false: Explain your reasoning or find an example or counterexample.

(a) If V is a subspace of $\mathbb{R}^3$ that does not contain any of the elementary column vectors e_1, e_2, e_3, then $V = \{\vec{0}\}$.

(b) If v_1, v_2, v_3, v_4 are linearly independent vectors, then v_1, v_2, v_3 are linearly independent.

(c) If v_1, v_2, v_3 are linearly independent vectors, then v_1, v_2, v_3, v_4 are linearly independent.

(d) It is possible for a 4×4 matrix A to have $\ker A = \operatorname{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ and

$$\operatorname{Im} A = \operatorname{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\rangle$$

(e) There exists a 4×4 matrix A with $\ker A = \operatorname{span}\langle e_1, e_2, e_3 \rangle$ and $\operatorname{Im} A = \operatorname{span}\langle e_3 + e_4 \rangle$

(f) There exists a 5×5 matrix A with $\ker A = \operatorname{Im} A$.

(g) There exists a 4×4 matrix A with $\ker A = \text{Im} A$.

(a) This is false. For instance, $V = \text{span}\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rangle$ is not the zero subspace but doesn't contain any of e_1, e_2, e_3.

(b) This is true. Since v_1, v_2, v_3, v_4 are linearly independent, the only solution (x_1, x_2, x_3, x_4) to $x_1v_1+x_2v_2+x_3v_3+x_4v_4 = \vec{0}$ is $(0, 0, 0, 0)$. Therefore, the only solution to $x_1v_1+x_2v_2+x_3v_3 = \vec{0}$ is $(0, 0, 0)$, so v_1, v_2, v_3 are linearly independent.

(c) This is false. For instance, let $v_1 = e_1, v_2 = e_2, v_3 = e_3$ in $\mathbb{R}^3$, and let v_4 be any vector in $\mathbb{R}^3$.

(d) False. Suppose A was a matrix satisfying the conditions. Notice that the two vectors spanning $\ker A$ are linearly independent, so $\dim \ker A = 2$. Similarly, we show by RREF or inspection that the three vectors spanning $\text{Im}A$ are linearly independent, so $\dim \text{Im}A = 3$. So $\dim \ker A + \dim \text{Im}A = 5$, but this contradicts the rank-nullity theorem, so no such matrix A exists.

(e) Yes. Notice that in this case $\dim \ker A + \dim \text{Im}A = 3 + 1 = 4$, so there is no issue from rank nullity. Since $e_1, e_2, e_3 \in \ker A$, the first three columns of A must be zero, and so we find the following example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f) False. A 5×5 matrix must have $\dim \ker A + \dim \text{Im}A = 5$ by rank-nullity, and since 5 is not an even number, we cannot have $\dim \ker A = \dim \text{Im}A$.

(g) True. Notice that by rank nullity, we must have $\dim \ker A = \dim \text{Im}A = 2$, so suppose A is a 4×4 matrix with $\ker A = \text{span}\langle e_1, e_2 \rangle$. This implies its first two columns are the zero vector. In order for the image of A to also be $\text{span}\langle e_1, e_2 \rangle$, the last two columns must span $\text{span}\langle e_1, e_2 \rangle$, so the following matrix works:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 3. Let $A : \mathbb{R}^8 \to \mathbb{R}^7$ be given by the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $\dim \ker A$ (hint: use rank-nullity and find $\dim \mathrm{Im} A$).

$\mathrm{Im}A$ is two dimensional by observation or by noticing that the row reduced echelon form has exactly two leading ones. Therefore, since $\dim \ker A + \dim \operatorname{im} A = 8$ and $\dim \operatorname{im} A = 2$, $\dim \ker A = 6$.

Exercise 4. Find a basis for the following subspaces of $\mathbb{R}^3$:

(a) $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right)$

(b) $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;\middle|\; v_1 - 3v_2 = 0 \right\}$

(c) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be any linear transformation such that $\dim \ker A = 0$. Find a basis for $V = \mathrm{Im} A$.

(a) $V = \operatorname{im} A$ for $A = \begin{bmatrix} 3 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -4 \end{bmatrix}$. RREF of A has leading ones in 1,2, and 4th column, so the first, second, and fourth of these vectors form a basis for V.

(b) Notice that $V = \ker A$ for $A = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}$. A is already row reduced, so writing the solutions to $A\vec{x} = \vec{0}$, we find that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ must satisfy

$$x_1 - 3x_2 = 0 \iff x_1 = 3x_2$$

Since only the first column of (the row reduced echelon form) of A has a leading one, x_2, x_3 are free, so

$$\ker A = \left\{ \begin{bmatrix} 3x_2 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \right\} = \left\{ x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \right\} = \operatorname{span}\langle \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$$

so a basis for $\ker A$ is $\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$

(c) Since $\dim \ker A = 0$, by rank nullity, $\dim \operatorname{Im} A = 3$. Therefore, since $\dim \operatorname{Im} A = 3$ and is a subspace of $\mathbb{R}^3$, $\operatorname{Im} A = \mathbb{R}^3$. Thus, a basis for A is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 5. For what values of $\lambda \in \mathbb{R}$ are the following pairs of vectors orthogonal?

(a) $\begin{bmatrix} \lambda \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} \lambda \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -\lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

(a) Orthogonal if and only if $-2\lambda + 3 = 0$, so orthogonal when $\lambda = 3/2$ and not otherwise.

(b) The dot product between these two vectors is 0, so orthogonal regardless of λ.

(c) Orthogonal if and only if $\lambda + \lambda = 0$, so only orthogonal when $\lambda = 0$.

Exercise 6. Consider two subspaces V and W of $\mathbb{R}^n$, where V is contained in W, denoted $V \subseteq W$.

(a) Show that $\dim(V) \leq \dim(W)$.

(b) Show that if $\dim(V) = \dim(W)$, then $V = W$.

(a) Let $\mathscr{B} = \{v_1, v_2, \ldots, v_k\}$ be a basis for V. Since $V \subset W$, each of the vectors $v_1, v_2, \ldots, v_k$ are in W, and thus $\mathscr{B}$ is a linearly independent subset of W. Therefore, we can choose vectors $v_{k+1}, \ldots, v_l \in W$ such that $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_l\}$ forms a basis for W, so $\dim W = l \geq k = \dim V$.

(b) Let $\mathscr{B} = \{v_1, v_2, \ldots, v_k\}$ be a basis for V and since $V \subset W$, each of the vectors $v_1, \ldots, v_k$ are in W, so $\mathscr{B}$ is a linearly independent subset of W. Since $\dim W = \dim V = k$ by assumption, $\mathscr{B}$ is thus a basis for V (see Thereom 3.3.4 in Bretscher).